Linear Algebra

UNIT-2

VECTOR SPACES

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Vector Space

- Real vector space V is a non-empty set of objects called vectors satisfying the following axioms

1. If $u, v \in V$ then $u+v \in V \Rightarrow V$ is closed under vector addition

2. If $c \in R$ and $u \in V$, then $cu \in V \Rightarrow V$ is closed under scalar multiplication

PROPERTIES

1. $u+v = v+u$ commutative

2. $u+(v+w) = (u+v)+w$ associative

3. $0+u = u+0 = u$ additive identity

4. For each u there is a unique vector $-u$ such that $u+(-u) = (-u)+u = 0$ inverse

5. $c_1(u+v) = c_1 u + c_1 v$

6. $(c_1+c_2)u = c_1 u + c_2 u$

distributive

7. $1u = u$ multiplicative identity

Examples of vector spaces

1. Set of all real no.s $\mathbb{R}$ associated with addition and scalar multiplication of real no.s (closed under set of all real no.s)

2. Set of all complex no.s $\mathbb{C}$ associated with addition and scalar multiplication of complex no.s

3. Set of all polynomials $R_n(x)$ with real coefficients associated with addition and multiplication of polynomials

4. Set of all vectors of dimencion n written as $\mathbb{R}^n$ associated with the addition and scalar multiplication as defined for n-dimensional vectors

5. Set of all matrices of dimension $m \times n$ associated with addition and scalar multiplication as defined for matrices

Q1. Prove that the set of all 2x2 matrices associated with the addition and scalar multiplication of 2x2 matrices is a vector space

1) Addition

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times2}$, $A' = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}_{2\times2}$ s.t $A, A' \in V_{2\times2}$

$$A + A' = \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix}_{2\times2} \longrightarrow$$ closed under 2x2 matrices

2) Scalar multiplication

Let $r =$ constant (scalar)

$$rA = \begin{bmatrix} ra & rb \\ rc & rd \end{bmatrix}_{2\times 2}, \quad rA \in V_{2\times 2}$$

3) Commutativity

$$A + A' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix}$$

$$= \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A' + A$$

4) Associativity

$$(A + A') + A'' = A + (A' + A'')$$

$$\begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix} + \begin{bmatrix} a'' & b'' \\ c'' & d'' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a'+a'' & b'+b'' \\ c'+c'' & d'+d'' \end{bmatrix}$$

$$\begin{bmatrix} a+a'+a'' & b+b'+b'' \\ c+c'+c'' & d+d'+d'' \end{bmatrix} = \begin{bmatrix} a+a'+a'' & b+b'+b'' \\ c+c'+c'' & d+d'+d'' \end{bmatrix}$$

5) Multiplicative associativity

$$r\left(s\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = s\left(r\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

$$r\begin{bmatrix} sa & sb \\ sc & sd \end{bmatrix} = s\begin{bmatrix} ra & rb \\ rc & rd \end{bmatrix}$$

$$\begin{bmatrix} rsa & rsb \\ rsc & rsd \end{bmatrix} = \begin{bmatrix} rsa & rsb \\ rsc & rsd \end{bmatrix}$$

6) Zero vector

$$A + 0 = A$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

7) Negative vector

$$A + -A = 0$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} = \begin{bmatrix} a-a & b-b \\ c-c & d-d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

8) Distributivity of sum of matrices

$$r\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\right) = r\begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix}$$

$$= \begin{bmatrix} ra+ra' & rb+rb' \\ rc+rc' & rd+rd' \end{bmatrix} = r\begin{bmatrix} a & b \\ c & d \end{bmatrix} + r\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$$

9) Distributivity of sums of scalars

$$(r+s)\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ra+sa & rb+sb \\ rc+sc & rd+sd \end{bmatrix}$$

$$= r\begin{bmatrix} a & b \\ c & d \end{bmatrix} + s\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

10) Multiplication by 1

$$1\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Q2. Show that the set of all polynomials of degree $n \le 3$ associated with addition and scalar multiplication form a vector space

the addition of polynomials of degree ≤ 3 results in a polynomial of degree ≤ 3

the multiplication of polynomials of degree ≤ 3 with scalar results in polynomial of degree ≤ 3

$\therefore$ remaining 8 conditions also satisfy

Q3. Show that the set of integers associated with addition and multiplication by a real number is not a vector space

addition of int with real no. is not always an int

eg: $2+\sqrt{5} = 2+\sqrt{5} \notin Z$

Q4. Verify whether V is a vector space

$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} ; x \geq 0,\ y \geq 0,\ x, y \in r \right\}$$

1) closure – holds good

2) associative – holds good

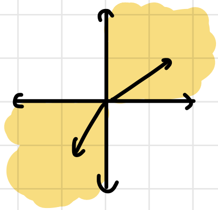
3) zero vector – does not hold

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\notin V$

Q5. Verify if W is a vector space

$$W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} ; xy \geq 0 \right\}$$



$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -5 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix} \notin W$$

$\therefore$ not a vector space

Subspace

A nonempty subset of a vector space is called a subspace of V if it itself is a vector space under the same operations (addition, scalar multiplication) as defined in a vector space

example

1) $0 \in W$ (zero vector always belongs to a subspace)

2) if $u, v \in W$, $u+v \in W$

3) if c is scalar and $u \in W$, $cu \in W$

- If U and W are two subspaces of a vector space V, then $U \cap W$ is also a subspace of V
- $0 \in U$ and $0 \in W$ (since they are subspaces)
- Intersection of any number of subspaces of V is a subspace of V

Echelon Form & Row-Reduced Echelon Form

unit 1, pg 4

Pivot Variables and Free Variables

$$Rx = 0$$

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{3\times 4} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix}_{4\times 1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- columns 1 & 3 have pivots $\Rightarrow$ u & w are pivot variables
- columns 2 & 4 have no pivots $\Rightarrow$ v & y are free variables

RANK OF A MATRIX

- no. of nonzero rows in echelon form U of A, denoted by $\rho(A)$ or r
- unit 1, pg 11

Q6. Find (i) echelon form U
(ii) RR echelon form R
(iii) Rank

$$A = \begin{bmatrix} 1 & 3 \\ -2 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 2R_1} \begin{bmatrix} 1 & 3 \\ 0 & 8 \end{bmatrix} = U$$

$R_2 \to \frac{1}{8} R_2$

rank $\rho(A) = 2$

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 3R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = R$$

Q7. Find (i) echelon form U
(ii) RR echelon form R
(iii) Rank

$$A = \begin{bmatrix} 2 & 6 & -2 \\ 3 & -2 & 8 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 3/2 R_1} \begin{bmatrix} 2 & 6 & -2 \\ 0 & -11 & 11 \end{bmatrix} = U$$

$R_2 \to -1/_{11} R_2$ | $R_1 \to 1/2 R_1$

rank $= 2$

$$R = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} \xleftarrow{R_1 \to R_1 - 3R_2} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

Q8. Find (i) echelon form U
(ii) RR echelon form R
(iii) Rank

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 4 & 1 & -2 \\ 6 & -1 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 2 & -2 & 4 \\ 0 & 5 & -10 \\ 0 & 5 & -10 \end{bmatrix}$$

rank $= 2$

$R_3 \to R_3 - R_2$

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \xleftarrow[R_1 \to R_1 + R_2]{} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \xleftarrow[R_2 \to 1/5 R_2]{R_1 \to 1/2 R_1} \begin{bmatrix} 2 & -2 & 4 \\ 0 & 5 & -10 \\ 0 & 0 & 0 \end{bmatrix}$$

Q9. Find (i) echelon form U
(ii) RR echelon form R
(iii) Rank

$$A = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 + 1/2 R_1]{R_2 \to R_2 + 3/2 R_1} \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} = U \xrightarrow{R_1 \to -1/2 R_1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\rho(A) = 1$$

Q10. Find (i) echelon form U
(ii) RR echelon form R
(iii) Rank

$$A = \begin{bmatrix} -2 & 3 & 1 \end{bmatrix} = U \xrightarrow{R_1 \to -1/2 R_1} \begin{bmatrix} 1 & -3/2 & -1/2 \end{bmatrix}$$

$$\text{rank} = \rho(A) = 1$$

Q11. Solve the following system of LE by identifying pivot and free variables.

$$x + 2y + 3z = 9$$
$$2x - 2z = -2$$
$$3x + 2y + z = 7$$

$$A = \left[\begin{array}{ccc:c} 1 & 2 & 3 & 9 \\ 2 & 0 & -2 & -2 \\ 3 & 2 & 1 & 7 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc:c} 1 & 2 & 3 & 9 \\ 0 & -4 & -8 & -20 \\ 0 & -4 & -8 & -20 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$\rho(A) = \rho(A:b) = 2 < 3$

$\therefore$ consistent, infinite solutions

$$\left[\begin{array}{ccc:c} 1 & 2 & 3 & 9 \\ 0 & -4 & -8 & -20 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

free : z

free variables: z
pivot variables: x, y

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -20 \\ 0 \end{bmatrix}$$

$$x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 2 \\ -4 \\ 0 \end{bmatrix} + z\begin{bmatrix} 3 \\ -8 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ -20 \\ 0 \end{bmatrix}$$

Let $z = k$ (free variables)

$-4y - 8k = -20$

$y + 2k = 5$

$y = 5 - 2k$

$x + 2(5 - 2k) + 3k = 9$

$x + 10 - 4k + 3k = 9$

$x = k - 1$

$(x, y, z) = (k-1, 5-2k, k)$, $k \in R$

Q12. Reduce $A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{bmatrix}$ to echelon form

and hence find the special solutions

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & c-1 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3}$$

$$U = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & c-1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

if $c=1$, $U = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$

solving $Rx = 0$

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

pivot: x

free: y, z, t

$$x + y + 2z + 2t = 0$$

$$\boxed{x = -y - 2z - 2t}$$

if $t=0$, $y=1$, $z=0$

$$\begin{bmatrix} -y-2z-2t \\ y \\ z \\ t \end{bmatrix} = y\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + z\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Q14. Reduce $A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$ to echelon form, RR echelon form

and hence find the special solutions

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 6 \end{bmatrix}$$

$$\downarrow R_3 \to R_3 - 2R_2$$

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xleftarrow{R_2 \to \frac{1}{3}R_2} \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

$$\xrightarrow{R_1 \to R_1 - 3R_2} \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

$$Rx = 0$$

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

free: y, t
pivot: x, z

$$x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$z + t = 0$$
$$z = -t$$

$$x + 3y + 3z + 2t = 0$$
$$x + 3y - 3t + 2t = 0$$
$$x = t - 3y$$

if $t = 0$, $z = 0$, $x = -3y$

$$\begin{bmatrix} t - 3y \\ y \\ -t \\ t \end{bmatrix} = y \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

special solutions

Linear COMBINATION

If V is a vector space and $v_1, v_2 \dots v_n$ are vectors in V, then $c_1v_1 + c_2v_2 + \dots + c_nv_n$ is a linear combination of the vectors ($c_1, c_2 \dots c_n$ are scalars)

LINEAR INDEPENDENCE of VECTORS

- if the only linear combination of vectors that produces a zero vector is the trivial solution, the vectors are linearly independent
- trivial combination: $0v_1 + 0v_2 + \dots + 0v_n = 0$
- if there exist other nonzero scalars $(c_1, c_2 \dots c_n)$ where at least some constants are nonzero and the linear combination of the vectors produces the zero vector, the vectors are linearly dependent
- vectors are independent if null space = zero vector

Check for dependence

1. Place vectors $\{v_1, v_2 \dots v_n\}$ as columns of matrix A
2. Apply Gaussian elimination on A
3. If pivot exists for every column, the vectors are linearly independent $\text{rank}(A) = n$
4. Else, linearly dependent

Q15. Check if $\begin{bmatrix}1\\2\\1\\3\end{bmatrix}, \begin{bmatrix}1\\1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\3\\7\\1\end{bmatrix}$ are independent in $\mathbb{R}^3$

$$A = \begin{bmatrix}1 & 1 & 1\\2 & 1 & 3\\1 & 2 & 7\\3 & 3 & 1\end{bmatrix} \xrightarrow[\substack{R_3 \to R_3 - R_1 \\ R_4 \to R_4 - 3R_1}]{R_2 \to R_2 - 2R_1} \begin{bmatrix}1 & 1 & 1\\0 & -1 & 1\\0 & 1 & 6\\0 & 0 & -2\end{bmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix}1 & 1 & 1\\0 & -1 & 1\\0 & 0 & 7\\0 & 0 & -2\end{bmatrix}$$

$$\xrightarrow{R_4 \to R_4 + 2/7 R_3} \begin{bmatrix}1 & 1 & 1\\0 & -1 & 1\\0 & 0 & 7\\0 & 0 & 0\end{bmatrix}$$

$\rho(A) = n = 3$

$\boxed{\therefore \text{ linearly independent}}$

Q16. Check whether $\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}3\\1\\0\end{bmatrix}, \begin{bmatrix}4\\3\\1\end{bmatrix}$ are independent in $\mathbb{R}^3$

$$A = \begin{bmatrix}1 & 3 & 4\\2 & 1 & 3\\1 & 0 & 1\end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix}1 & 3 & 4\\0 & -5 & -5\\0 & -3 & -3\end{bmatrix} \xrightarrow{R_3 \to R_3 - 3/5 R_2} \begin{bmatrix}1 & 3 & 4\\0 & -5 & -5\\0 & 0 & 0\end{bmatrix}$$

$\therefore$ linearly dependent

$\rho(A) = 2, \quad n = 3$

note

1. Columns of invertible square matrix are always independent
2. Columns of matrix $A_{m \times n}$ with $m < n$ are always dependent
3. Columns of A are independent when $N(A) = Z$ ← zero (null space)
4. The 'r' nonzero rows of echelon matrix U and reduced matrix R are always independent and so are the 'r' columns that contain the pivots

BASIS

A subset $S = \{v_1, v_2 \dots v_n\}$ of a vector space is called a basis for vector space V if

1. S is a linear independent set
2. S spans the vector space V
3. Every vector in the space can be represented as a linear combination of the basis vectors

The dimension of the vector space is the number of basis vectors

Basis is maximal independent set and minimal spanning set

Unique way to write a vector v as linear combination of basis vectors

examples of basis sets

1. Basis of $\mathbb{R}^2$ —— $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

2. Basis of $\mathbb{R}^3$ —— $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

3. Basis of 2×2 matrices —— $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

4. Basis of n- degree polynomial —— $\{1, t, t^2, \dots, t^n\}$

Q17. Decide dependence or independence of the following vectors

vectors: $(1,3,2)$, $(2,1,3)$, $(3,2,1)$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & -1 & -5 \end{bmatrix} \xrightarrow{R_3 \to R_3 - \frac{1}{5}R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & 0 & -18/5 \end{bmatrix}$$

$-5 + 7/5$

$\rho(A) = n = 3$

independent

Q18. Decide dependence or independence of the following vectors

(1, -3, 2), (2, 1, -3), (-3, 2, 1)

$$A = \begin{bmatrix} 1 & 2 & -3 \\ -3 & 1 & 2 \\ 2 & -3 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 + 3R_1} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & -7 & 7 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2 < n = 3$

$\therefore$ linearly dependent

Q19. Decide dependence or independence of the following vectors

(1, 1), (2, 3), (1, 2)

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$\rho(A) = 2 < n = 3$

$\therefore$ linearly dependent

Q20. If w_1, w_2, w_3 are independent vectors. Show that the differences

$$v_1 = w_2 - w_3$$
$$v_2 = w_1 - w_3 \quad \text{are dependent}$$
$$v_3 = w_1 - w_2$$

w_1, w_2, w_3 are independent

$$\alpha w_1 + \beta w_2 + \gamma w_3 = 0 \Rightarrow \alpha = \beta = \gamma = 0$$

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$
$$c_1(w_2 - w_3) + c_2(w_1 - w_3) + c_3(w_1 - w_2) = 0$$

$$w_1(c_2 + c_3) + w_2(c_1 - c_3) + w_3(-c_1 - c_2) = 0$$

$$c_2 + c_3 = 0 \Rightarrow c_3 = -c_2$$

$$c_1 - c_3 = 0 \Rightarrow c_1 = c_3 = -c_2$$

$$-c_1 - c_2 = 0 \Rightarrow c_1 = -c_2$$

$$\boxed{c_1 = -c_2, \quad c_2 = c_2, \quad c_3 = -c_2}$$

$\therefore$ c_1, c_2, c_3 can assume nonzero values also

$\therefore$ v_1, v_2, v_3 are linearly dependent

Q21. Find the basis and hence the dimension of subspaces of R^4

(i) All vectors whose components are equal
(ii) All vectors whose components add up to zero

(i) 4 components define dimension

$$V = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} \in R^4 \sim \begin{bmatrix} x \\ x \\ x \\ x \end{bmatrix}$$

$$V = \left\{ x \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} : x \in R \right\}$$ subspace

basis $= \{ [1 \ 1 \ 1 \ 1]^T \}$

$\therefore$ dimension $= 1$

(ii)

$$V = \begin{bmatrix} x \\ y \\ z \\ -x-y-z \end{bmatrix} \in \{x, y, z \in R\}$$

$$V = \left\{ x \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} : x, y, z \in R \right\}$$

basic: $\{ [1 \ 0 \ 0 \ -1]^T, [0 \ 1 \ 0 \ -1]^T, [0 \ 0 \ 1 \ -1]^T \}$

$\therefore$ dimension $= 3$

Q22. Let V be a subspace of 4D space R^4 such that

$$V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in R^4 ; \; x_1 - x_2 + x_3 - x_4 = 0 \right\}$$

Find basis and dimension

$$x_1 = x_2 - x_3 + x_4$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \sim \begin{bmatrix} x_2 - x_3 + x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$V = \left\{ x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} : x_2, x_3, x_4 \in R \right\}$$

$$\text{basis} = \{ [1 \; 1 \; 0 \; 0]^T, [-1 \; 0 \; 1 \; 0]^T, [1 \; 0 \; 0 \; 1]^T \}$$

dimensions = 3

Q23. Find a basis for each of the following subspaces of 2x2 matrices

(i) All diagonal matrices
(ii) All symmetric matrices ($A^T = A$)
(iii) All skew symmetric matrices ($A^T = -A$)

2x2 : $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$; $a, b, c, d \in R$

basis: $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

dimensions = 4

(i) $V = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = \left\{ a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} : a, d \in R \right\}$

basis: $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

dimensions = 2

(ii) $V = \begin{bmatrix} a & b \\ b & d \end{bmatrix} = \left\{ a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

basis: $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

dimensions = 3

(iii) $V = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} = \left\{ b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$

basis $= \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$ dimensions $= 1$

Four Fundamental Subspaces of a Matrix

1) $C(A)$ – column space
- subspace of R^m
- linear combinations of column vectors of A
- $\rho(A) = \text{dimensions}(C(A)) = k$
- basis of $C(A)$ corresponds to columns having a pivot in echelon form of A

2) $C(A^T)$ – row space
- subspace of R^n
- linear combination of rows of A
- $\rho(A) = \dim(C(A^T)) = k$
- basis of $C(A^T)$ is set of row vectors in A or in the echelon form, corresponding to the pivots in echelon form

3) $N(A)$ – null space
- all solutions of system $Ax = 0$
- subspace of R^n
- if $\rho(A) = k$ then $\dim(N(A)) = n-k$

4) $N(A^T)$ – left null spaces
- all solutions of system $A^T x = 0$
- subspace of R^n
- if $\rho(A) = k$ then $\dim(N(A^T)) = m-k$
- linear combination of rows that gives 0

1. The row space of Amxn is the column space of A^T.
   It is spanned by the rows of A.

2. The left null space contains all vectors y for which $A^T y = 0$.

3. N(A) and C(AT) are subspaces of R^m

4. N(A^T) and C(A) are subspaces of R^n

5. Dim C(A) = Dim C(AT) = r = rank of A

6. Dim N(A) = n-r and Dim N(AT) = m- r.
7. The dimension of the null space of a matrix is called its nullity.

$\boxed{\text{RANK-NULLITY THEOREM}}$

$A_{m \times n}$

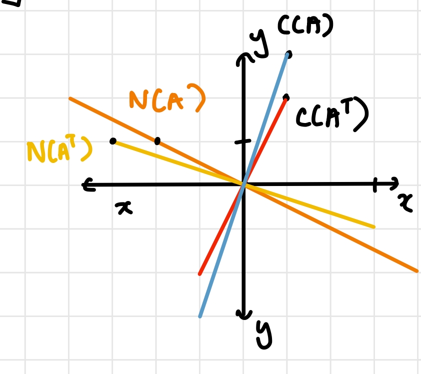
- $\dim C(A) + \dim N(A) = n = r + (n-r)$
- $\dim C(A^T) + \dim N(A^T) = m = r + (m-r)$

Q24. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

$m = 2, \ n = 2, \ r = 1$

1. C(A) is line through (1, 3)

2. C(A^T) is the line through (1, 2)



3. $N(A)$ is the line through $(-2,1)$

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc:c} 1 & 2 & 0 \\ 3 & 6 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - 3R_1} \left[\begin{array}{cc:c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x + 2y = 0$$

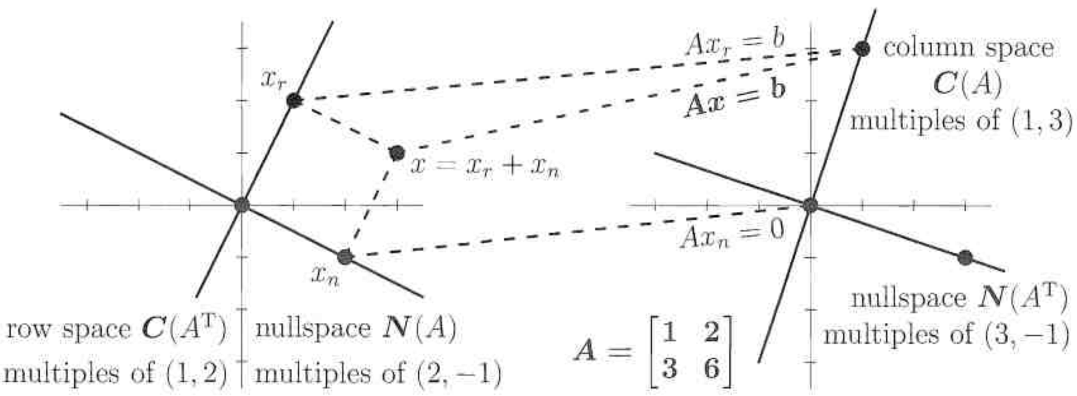
$$x = -2y$$

4. $N(A^T)$ is the line through $(-3,1)$

$$\left[\begin{array}{cc:c} 1 & 3 & 0 \\ 2 & 6 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{cc:c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x + 3y = 0$$

$$x = -3y$$



Q25. Find dimensions and basis for each of the four fundamental subspaces of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 3 & 6 & 3 & 7 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2 rows

pivot columns 1 & 4

$\rho(A) = 2$

$n = 4$

1. $C(A) = \left\{ c_1 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} \,\middle|\, c_1, c_2 \in \mathbb{R} \right\}$

basis: $\{ [1 \ 1 \ 3]^T, [2 \ 3 \ 7]^T \}$

dim = 2

2. $C(A^T) = \left\{ c_1 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix} \,\middle|\, c_1, c_2 \in \mathbb{R} \right\}$

basis = $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix} \right\}$, dim = 2

3. $N(A)$

$Ax = 0$

$$[U:0] \quad \left[\begin{array}{cccc:c} 1 & 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$t = 0$

$x + 2y + z = 0$

$x = -2y - z$

special

$$\begin{bmatrix} -2y-z \\ y \\ z \\ 0 \end{bmatrix} = y\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + z\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Let $y = c_1$ and $z = c_2$

$$N(A) = \left\{ c_1\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$\dim = 2$

4. $N(A^T)$

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 6 \\ 1 & 1 & 3 \\ 2 & 3 & 7 \end{bmatrix} \xrightarrow[R_4 \to R_4 - 2R_1]{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1}} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_4} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$y + z = 0$ $\qquad$ $x + y + 3z = 0$

let $y = c$ $\qquad$ $x + c - 3c = 0$

$z = -c$ $\qquad$ $x = 2c$

$= (2c, c, -c)$

$$N(A^T) = \left\{ c \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \;\middle|\; c \in R \right\}$$

$$\text{basis} = \left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\}$$

$\dim = 1$

Q26. Obtain $N(A^T)$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 6 & 3 \\ 0 & 2 & 5 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 6 & 2 \\ 1 & 3 & 5 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 1 & 5 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 1/2 R_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

$\rho(A^T) = 3$

no zero rows : $N(A^T) = \vec{0}$ (trivial solution)

$$N(A^T) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$ origin

$$\text{basis} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \dim = 0$$

Q27. Find $N(A^T)$

(a) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}$ (b) $B = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$

(a) $A^T = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 4 \\ 1 & 4 & 8 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 3 & 6 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$\rho(A) = 2$
$n = 3$

Let $z = k$

$y + 2k = 0$
$y = -2k$

$x - 2k + 2k = 0$
$x = 0$

$$= \begin{bmatrix} 0 \\ -2k \\ k \end{bmatrix} = k \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

$$N(A^T) = \left\{ k \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}, k \in \mathbb{R} \right\}$$

basis = $\{ [0 \ -2 \ 1]^T \}$, dim = 1

(b) $B = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \Rightarrow B^T = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & -3 \\ 3 & 9 & 3 \\ 2 & 7 & 4 \end{bmatrix}$ $\xrightarrow[\substack{R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 2R_1}]{R_2 \to R_2 - 3R_1}$ $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 3 & 6 \\ 0 & 3 & 6 \end{bmatrix}$

$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \\ 0 & 3 & 6 \end{bmatrix} \xrightarrow{R_4 \to R_4 - R_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Let $z = k$

$3y + 6k = 0 \qquad x - 4k - k = 0$

$y = -2k \qquad x = 5k$

$\begin{bmatrix} 5k \\ -2k \\ k \end{bmatrix} = k \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} \Rightarrow N(B^T) = \left\{ k \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}, k \in R \right\}$

basis $= \left\{ \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} \right\}$ dim $= 1$

Q28. Find vector space

(a) $c = [0]$ (b) $D = [0 \ -3]$ (c) $= \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

(a) $C = [0] = Z$ $[U : b] = [0 \ : \ 0]$

$C^T = [0]$

smallest vector space $C = Z = [0]$

span V

(b) $D = \begin{bmatrix} 0 & -3 \end{bmatrix}$

$\cdot$ $V =$ x-axis in R^1

(c) $= \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ $V =$ y-axis in R^1

Q29. Find column space & null space

$$F = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & 3 & 1 \\ 3 & 8 & 2 & 3 \end{bmatrix} \xrightarrow[R_3 \to R_3 + 3/2 R_1]{R_2 \to R_2 + R_1} \begin{bmatrix} -2 & 4 & -2 & -4 \\ 0 & -2 & 1 & -3 \\ 0 & 14 & -1 & -3 \end{bmatrix}$$

$$\big\downarrow R_3 \to R_3 + 7R_2$$

$$U = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 6 & -24 \end{bmatrix}$$

$$C(F) = \left\{ c_1 \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ -6 \\ 8 \end{bmatrix} + c_3 \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix} \,\middle|\, c_1, c_2, c_3 \in R \right\}, \quad \dim = 3$$

$N(F) \Rightarrow$

$6z - 24t = 0$

$z - 4t = 0$

Let $t = k$

$z = 4k$

$-2y + z - 3t = 0$

$-2y + 4k - 3k = 0$

$-2y + k = 0$

$y = k/2$

$-2x + 2k - 8k - 4k = 0$

$-x + k - 4k - 2k = 0$

$-x - 5k = 0$

$x = -5k$

$$N(F) = \left\{ c \begin{bmatrix} -10 \\ 1 \\ 8 \\ 2 \end{bmatrix}, \; c \in R \right\}, \quad \dim = 1$$

Q30. Find column space and null space. Give an example of a matrix whose col space is same as that of A but the null space is different.

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 7 \\ 5 & 3 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 5R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 0 \\ 0 & 7 \\ 0 & 3 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3/7 R_2} \begin{bmatrix} 1 & 0 \\ 0 & 7 \\ 0 & 0 \end{bmatrix} = U$$

column space: columns that have pivots

$$C(A) = \left\{ c_1 \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 7 \\ 3 \end{bmatrix} \middle| \ c_1, c_2 \in R \right\}$$

$$\text{Basis} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 3 \end{bmatrix} \right\}$$

$$\text{Dim}(A) = 2$$

$N(A) \Rightarrow \quad Ax = 0$

$$7y = 0 \qquad\qquad x = 0$$
$$y = 0$$

$$\underset{3\times2}{\begin{bmatrix} 1 & 0 \\ 0 & 7 \\ 0 & 0 \end{bmatrix}} \underset{2\times1}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{3\times1}{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

null space $N(A) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

origin

column space of B = C(B) = C(A)

$$B = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 7 & 14 \\ 5 & 3 & 6 \end{bmatrix}$$

OR

columns

$C_3 = C_1 + C_2$

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 7 & 9 \\ 5 & 3 & 8 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 5R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 7 & 7 \\ 0 & 3 & 3 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3/7 R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 7 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

C(B) = C(A)

null space: $Bx = 0$

$7y + 7z = 0$

$y + z = 0$

Let $z = k$

$y = -k$

$x + z = 0$

$x = -k$

$$N(B) = \left\{ k \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \mid k \in R \right\}$$

$$\text{Basis} = \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$\text{Dim}(B) = 1$$

Q31. Let $V = \{ (a, b, c, d) \mid b + c + d = 0 \}$ and

$W = \{ (a, b, c, d) \mid a + b = 0 \text{ \& } c = 2d \}$

be subspaces of R^4. Find $\text{Dim}(V \cap W)$

$$V = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}, \ b + c + d = 0 \right\}$$

$$d = -c - b$$

$$V = \left\{ \begin{bmatrix} a \\ b \\ c \\ -c-b \end{bmatrix} \right\} = \left\{ a \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

Dim = 3 , 3D plane in R^4

$$W = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}, \ a+b=0 \ \text{and} \ c=2d \right\}$$

$$b = -a \qquad c = 2d$$

$$W = \left\{ \begin{bmatrix} a \\ -a \\ 2d \\ d \end{bmatrix} \right\} = \left\{ a \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \ a, d \in \mathbb{R} \right\}$$

$$\text{Dim} = 2$$

W is 2D plane in $\mathbb{R}^4$

$$V \cap W \Rightarrow \quad b+c+d=0 \Rightarrow -a+2d+d=0 \Rightarrow a = 3d$$
$$a+b=0 \Rightarrow b=-a=-3d$$
$$c=2d$$

$$V \cap W = \left\{ \begin{bmatrix} 3d \\ -3d \\ 2d \\ d \end{bmatrix} \right\} = \left\{ d \begin{bmatrix} 3 \\ -3 \\ 2 \\ 1 \end{bmatrix}, \ d \in \mathbb{R} \right\}$$

$$\text{Dim}(V \cap W) = 1$$

$V \cap W$ is a 1D line in $\mathbb{R}^4$ spanned by $(3, -3, 2, 1)$

Q32. Find C(A) and N(A) for the following

(i) $A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & 3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 3 & 5 & -1 & 6 \\ 2 & 4 & 1 & 2 \\ 2 & 0 & -7 & 11 \end{bmatrix}$

(i) $$A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & 3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3/2 R_1]{R_2 \to R_2 + R_1} \begin{bmatrix} -2 & 4 & -2 & 4 \\ 0 & -2 & 1 & -3 \\ 0 & 2 & 5 & 3 \end{bmatrix}$$

$$\Big\downarrow R_3 \to R_3 + R_2$$

$$U = \begin{bmatrix} -2 & 4 & -2 & 4 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 6 & 0 \end{bmatrix}$$

$$C(A) = \left\{ c_1 \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ -6 \\ 8 \end{bmatrix} + c_3 \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix} \right\}$$

$$\text{Basis} = \left\{ \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 8 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix} \right\}$$

Dim = 3

$N(A) \Rightarrow \quad Ax = 0 \quad$ or $\quad Ux = 0$

$6z = 0 \qquad -2y - 3t = 0 \qquad -2x - 6k + 4k = 0$

$z = 0 \qquad$ Let $t = k \qquad -2x - 2k = 0$

$-2y = 3k \qquad x = -k$

$y = \frac{-3}{2}k$

$$N(A) = \left\{ k \begin{bmatrix} -2 \\ -3 \\ 0 \\ 2 \end{bmatrix}, \; k \in R \right\}$$

$\dim = 1 \qquad$ Basis $= \left\{ \begin{bmatrix} -2 \\ -3 \\ 0 \\ 2 \end{bmatrix} \right\}$

Q33. Reduce to RREF and determine ranks. Identify free & pivot variables.

(i) $A = \begin{bmatrix} 2 & 4 & -3 & -2 \\ -2 & -3 & 2 & -5 \\ 1 & 3 & -2 & 2 \end{bmatrix}$

(ii) $B = \begin{bmatrix} 0 & 2 & 3 & 7 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 4 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(i) $A = \begin{bmatrix} 2 & 4 & -3 & -2 \\ -2 & -3 & 2 & -5 \\ 1 & 3 & -2 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_3 - \frac{1}{2}R_1]{R_2 \to R_2 + R_1} \begin{bmatrix} 2 & 4 & -3 & -2 \\ 0 & 1 & -1 & -7 \\ 0 & 1 & -1/2 & 3 \end{bmatrix}$

$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 2 & 4 & -3 & -2 \\ 0 & 1 & -1 & -7 \\ 0 & 0 & 1/2 & 10 \end{bmatrix} \xrightarrow[R_3 \to 2R_3]{R_1 \to \frac{1}{2}R_1} \begin{bmatrix} 1 & 2 & -3/2 & -1 \\ 0 & 1 & -1 & -7 \\ 0 & 0 & 1 & 20 \end{bmatrix}$

$\xrightarrow[R_2 \to R_2 + R_3]{R_1 \to R_1 + \frac{3}{2}R_3} \begin{bmatrix} 1 & 2 & 0 & 29 \\ 0 & 1 & 0 & 13 \\ 0 & 0 & 1 & 20 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 13 \\ 0 & 0 & 1 & 20 \end{bmatrix}$

rank = 3   free variables = t pivot variables = x, y, z

(ii) $B = \begin{bmatrix} 0 & 2 & 3 & 7 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 4 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 7 \\ 1 & 3 & 4 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 7 \\ 0 & 2 & 3 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 7 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{2}R_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3/2 & 7/2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\xrightarrow{R_1 \to R_1 - R_3}$

rank = 3

free variables = z

pivot variables = x, y, t

$$\Big\downarrow R_2 \to R_2 - 7/2 R_3$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Q34. Examine if following vectors are linearly independent

$$\left\{ \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ -4 & 4 \end{bmatrix} \right\} \text{ in } M_{2\times 2}(\mathbb{R})$$

linear combination that produces 0 vector

$$c_1 \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} + c_3 \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 2 & 1 \\ -4 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 & 0 \\ -2c_1 & c_1 \end{bmatrix} + \begin{bmatrix} 0 & -c_2 \\ c_2 & c_2 \end{bmatrix} + \begin{bmatrix} -c_3 & 2c_3 \\ c_3 & 0 \end{bmatrix} + \begin{bmatrix} 2c_4 & c_4 \\ -4c_4 & 4c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 - c_3 + 2c_4 & -c_2 + 2c_3 + c_4 \\ -2c_1 + c_2 + c_3 - 4c_4 & c_1 + c_2 + 4c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned} c_1 + 0c_2 - c_3 + 2c_4 &= 0 \\ 0c_1 - c_2 + 2c_3 + c_4 &= 0 \\ -2c_1 + c_2 + c_3 - 4c_4 &= 0 \\ c_1 + c_2 + 0c_3 + 4c_4 &= 0 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & 2 & 1 \\ -2 & 1 & 1 & -4 \\ 1 & 1 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_4 \to R_4 - R_1$ ↓ $R_3 \to R_3 + 2R_1$

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & 2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 2 \end{bmatrix} \xrightarrow[R_4 \to R_4 + R_2]{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

↓ $R_4 \to R_4 - 3R_3$

rank = 3 < 4
1 free, 3 pivot

$$U = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

∴ vectors are linearly dependent

Q35. Examine if following vectors are linearly independent

$\{t^2+t+2,\ 2t^2+t,\ 3t^2+2t+2\}$

$$\begin{array}{c c} t^2 & \\ t & A= \\ 1 & \end{array} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 2 & 0 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 0 & -4 & -4 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 4R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

rank = 2 < n = 3

can also do (2,1,1)

∴ linearly dependent

existence of inverses

EXISTENCE OF INVERSE FOR RECTANGULAR MATRIX

- Let $A_{m \times n}$ be a matrix such that $\rho(A) = m$ $(m \leq n)$
- Then $Ax = b$ has at least one solution x for every b if and only if the columns span R^m

as many

- In this case, A has a right inverse C such that $AC = I_{m \times m}$

$$\boxed{A_{m \times n} C_{n \times m} = I_{m \times m}}$$

- Let $A_{m \times n}$ be a matrix such that $\rho(A) = n$ $(n \leq m)$
- Then $Ax = b$ has at most one solution x for every b if and only if the columns are linearly independent

unique

- In this case, A has a left inverse B such that $BA = I_{n \times n}$

$$\boxed{B_{n \times m} A_{m \times n} = I_{n \times n}}$$

Best Right Inverse

$$C = A^T (AA^T)^{-1}$$

Best Left Inverse

$$B = (A^T A)^{-1} A^T$$

Q36. Find right inverse of A

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}_{2\times 3}$$

$m=2 \quad n=3$

$$A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \\ a & b \end{bmatrix}_{3\times 2} = C \leftarrow \text{infinitely many}$$

$$AC = I_{2\times 2}$$

Q37. Find left/right inverse for the matrix (whichever exists)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Step 1: Apply GA and obtain rank

Step 2: Check if $\rho = m$ or $\rho = n$

Step 3: if $\rho(A) = m$ then RI exists

$$RI = A^T (AA^T)^{-1}$$

if $\rho(A) = n$ then LI exists

$$LI = (A^T A)^{-1} A^T$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{2\times 3}$$

$$\rho = 2 = m \Rightarrow \text{RI exists}$$

$$RI = C = A^T (A A^T)^{-1}$$

inverse of 2x2 matrix A

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A \cdot A^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$(A \cdot A^T)^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \left(\frac{1}{3}\right)$$

$$A^T (A \cdot A^T)^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}_{3\times 2} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}_{2\times 2} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ -1 & 2 \end{bmatrix}_{3\times 2}$$

$$C = \begin{bmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix}$$

Q38. Find the inverse of A

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$m = 3 \qquad n = 2$

$\rho = 2 = n \Rightarrow$ left inverse exists

$$B = (A^T \cdot A)^{-1} A^T$$

$$A^T \cdot A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$(A^T \cdot A)^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \left(\frac{1}{3}\right)$$

$$(A^T \cdot A)^{-1} \cdot A^T = \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}_{2\times 2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{2\times 3}$$

$$B = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}_{2\times 3}$$

MATRICES of RANK 1

- every row is a multiple of the first row
- can write matrix as the product of a column vector and a row vector
- $A = uv^T$

Q39. $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ -1 & -2 & -3 & -4 \end{bmatrix}$ as column × row

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ -1 & -2 & -3 & -4 \end{bmatrix}$$

Q40. Which vectors (b_1, b_2, b_3) are in $C(A)$? Which combination of rows of A gives 0? → $N(A^T)$

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_1} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C(A) = \left\{ c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; c_1, c_2 \in \mathbb{R} \right\}$$

$$\text{Basis} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad \text{Dim}(C(A)) = 2$$

$N(A^T) \Rightarrow$ use $[A : b]$

$$[A : b] = \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & b_1 \\ 0 & 0 & 0 & 0 & b_2 \\ 2 & 4 & 0 & 1 & b_3 \end{array}\right]$$

$$\Big\downarrow R_3 \to R_3 - 2R_1$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & b_1 \\ 0 & 0 & 0 & 0 & b_2 \\ 0 & 0 & 0 & -5 & b_3 - 2b_1 \end{array}\right]$$

$$\Big\downarrow R_3 \leftrightarrow R_2$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & b_1 \\ 0 & 0 & 0 & -5 & b_3 - 2b_1 \\ 0 & 0 & 0 & 0 & b_2 \end{array}\right]$$

$N(A^T)$ = linear combination of zero rows

$$N(A^T) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad b_2 = 0$$

Q41. Vectors $(1,4,2)$, $(2,5,1)$, $(3,6,0)$

Let $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Find $N(A^T)$, $C(A^T)$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 1 & 0 \end{bmatrix} \quad \text{from vectors}$$

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 4 & 5 & 6 & b_2 \\ 2 & 1 & 0 & b_3 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 4R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & -3 & -6 & b_3 - 2b_1 \end{array}\right]$$

$$\downarrow R_3 \to R_3 - R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & 0 & 0 & b_3 - b_2 + 2b_1 \end{array}\right]$$

$$b_3 - b_2 + 2b_1 = 0$$

$$2b_1 - b_2 + b_3 = 0$$

$$N(A^T) = \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\} \qquad \mathrm{Dim}(N(A^T)) = 1$$

$$C(A^T) = \left\{ c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \;\middle|\; c_1, c_2 \in R \right\}$$

$$\text{Basis} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\} \qquad \text{Dim}(C(A^T)) = 2$$

Q42. Reduce the following to RREC and determine their ranks

(a) Identify pivot & free vars

(b) Find special solutions to $Ax = 0$

(i) $A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & 3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 0 & 2 & 3 & 7 \\ 1 & 1 & 1 & 1 \\ 1 & 4 & 4 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(i) $A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & 3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3/2 R_1]{R_2 \to R_2 + R_1} \begin{bmatrix} -2 & 4 & -2 & -4 \\ 0 & -2 & 1 & -3 \\ 0 & 2 & 5 & 3 \end{bmatrix}$

$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} -2 & 4 & -2 & -4 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 6 & 0 \end{bmatrix} \xrightarrow[R_2 \to -\frac{1}{2} R_2]{R_1 \to -\frac{1}{2} R_1} \begin{bmatrix} 1 & -2 & 1 & 2 \\ 0 & 1 & -1/2 & 3/2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$R_3 \to \frac{1}{6} R_3$

$\xrightarrow[R_1 \to R_1 - R_3]{R_2 \to R_2 + 1/2 R_3} \begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow[R_1 \to R_1 + 2R_2]{} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & 1 & 0 \end{bmatrix} = R$

pivots: x, y, z

free: t

$N(A) \Rightarrow$ $\boxed{z = 0}$

$y + \frac{3}{2} t = 0$

$\boxed{y = -3/2 t}$

$x \quad + 3t \quad + 2t = 0$

$\boxed{x = -5t}$

$N(A) = \left\{ t \begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix} \middle| \, t \in R \right\}$

$N(A) = \left\{ t \begin{bmatrix} -10 \\ -3 \\ 0 \\ 2 \end{bmatrix} \middle| \, t \in R \right\}$

(ii) $A = \begin{bmatrix} 0 & 2 & 3 & 7 \\ 1 & 1 & 1 & 1 \\ 1 & 4 & 4 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 7 \\ 1 & 4 & 4 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 7 \\ 0 & 3 & 3 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3/2 R_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 7 \\ 0 & 0 & -3/2 & -7/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\xrightarrow{R_3 \to -2/3 R_3 \;\mid\; R_2 \to 1/2 R_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3/2 & 7/2 \\ 0 & 0 & 1 & 7/3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2 - 7/2 R_4 \\ R_1 \to R_1 - R_4}]{R_3 \to R_3 - 7/3 R_4} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\xrightarrow{R_1 \to R_1 - R_3 \;\mid\; R_2 \to R_2 - 3/2 R_3} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_{4\times4}$

pivots: x, y, z, t
no free

$N(A^T) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ (origin)

Q43. For which vectors $b = (a \ b \ c)$ does the following system $Ax = b$ has a solution

$$x + 2y = a$$
$$x + y + 2z = b$$
$$3x - 4z = c$$

$$[A:b] = \left[\begin{array}{ccc:c} 1 & 2 & 0 & a \\ 1 & 1 & 2 & b \\ 3 & 0 & -4 & c \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc:c} 1 & 2 & 0 & a \\ 0 & -1 & 2 & b-a \\ 0 & -6 & -4 & c-3a \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 6R_2} \left[\begin{array}{ccc:c} 1 & 2 & 0 & a \\ 0 & -1 & 2 & b-a \\ 0 & 0 & -16 & 3a-6b+c \end{array}\right]$$

$$\rho(A) = 3 = n$$

consistent, unique

for all values of a, b, c solution exists

Q44. Let $A = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix}$ $\quad w = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Determine if w is in $C(A)$, $N(A)$

$$A = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 1/2 R_1} \begin{bmatrix} -6 & 12 \\ 0 & 0 \end{bmatrix}$$

$$C(A) = \left\{ c_1 \begin{bmatrix} -6 \\ -3 \end{bmatrix}, \ c_1 \in R \right\}$$

for $c_1 = -1/3$, $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = w$

$\therefore w \in C(A)$

$N(A) \Rightarrow$ $$-6x + 12y = 0$$

$$x = 2y$$

$$N(A) = \left\{ \begin{bmatrix} 2y \\ y \end{bmatrix} \right\}$$

$$= \left\{ y \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

$$N(A) = \left\{ k \begin{bmatrix} 2 \\ 1 \end{bmatrix} \mid k \in R \right\}$$

for $k = 1$, $V = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = w$

$\therefore w \in N(A)$